DESIGN INVESTIGATION OF COMBINATIONS OF SPHERICAL AND CYLINDRICAL SHELLS UNDER HYDROSTATIC PRESSURE

Gerald Edwin Van Sickle

Thesis Supervisor: J. Harvey Evens
Title: Professor of Naval Architecture

June 1967

DESIGN INVESTIGATION OF COMBINATIONS OF SPHERICAL AND CYLINDRICAL SHELLS UNDER HYDROSTATIC PRESSURE

by

Gerald Edwin Van Sickle

Met. E., Colorado School of Mines
(1958)

SUBMITTED IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE DEGREES OF

MASTER OF SCIENCE AND NAVAL ENGINEER

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June, 1967

Signature of Author_______________________________
Department of Naval Architecture
and Marine Engineering, June, 1967

Certified by_______________________________
Thesis Supervisor

Accepted by_______________________________
Chairman, Departmental Committee
on Graduate Students

ABSTRACT

DESIGN INVESTIGATION OF COMBINATIONS OF SPHERICAL AND CYLINDRICAL SHELLS UNDER HYDROSTATIC PRESSURE

Gerald Edwin Van Sickle

Lieutenant, United States Navy

Submitted to the Department of Naval Architecture and Marine Engineering on June 16, 1967, in partial fulfillment of the requirements for the Master of Science degree in Naval Architecture and Marine Engineering and the Professional degree, Naval Engineer.

Shells consisting of two spherical shell sections joined together by a cylindrical shell section and shells consisting of two intersecting spherical shell sections were considered in this investigation. The purpose of this investigation was to determine the shell of optimum proportions and configuration from all the shells considered and to evaluate the penalty incurred by using a shell of other than optimum proportions and configuration.

The shell of optimum proportions and configuration was defined as that shell with the lowest weight to displacement ratio, given the conditions that all shells are made of the same material and are designed to withstand the same hydrostatic pressure.

A relationship between the weight to displacement ratio of these shells and their geometric ratios was developed. The independent geometric ratios chosen for use in this investigation were: 1) the ratio of the radius of the larger spherical shell to the radius of the smaller spherical shell, 2) the ratio of the radius of the cylindrical shell to the radius of the smaller spherical shell, and 3) the ratio of the length of the cylindrical shell to the radius of the smaller spherical shell.

The geometric ratios were systematically and independently varied and the effect upon the weight displacement ratio was determined. The results were plotted graphically.

The shell of optimum proportions and configuration was found to be a shell consisting of two intersecting spherical shell sections with equal radii and with a center to center separation of the spherical shell sections equal to their radii. The weight to displacement ratio of this shell was found to be 11.1% less than that of a single spherical shell.

Thesis Supervisor: J. Harvey Evans
Title: Professor of Naval Architecture

TABLE OF CONTENTS

INTRODUCTION

Considerable interest has recently been shown in manned vehicles capable of exploring the depths of the ocean. The structural design of the hull of such a vehicle must consider strength to weight ratios of possible hull materials and the relation between the structural weight and the displacement of the hull.

For this reason, a strong possibility exists that these vehicles will utilize spherical shells either as the main pressure hull or to close off the ends of a basically cylindrical hull. There is also the possibility of the pressure hull being two spherical shells connected by a cylindrical section of relatively short length.

This work will study pressure hulls consisting of two spherical shell sections joined together by a cylindrical shell section. The length of the cylindrical section, the radius of the cylindrical section, and the radii of the two spherical sections will be varied. By this means hulls can be studied whose geometries range from 1) a single sphere, to 2) a cylinder with hemispherical ends, to 3) a dumbbell shape, to 4) two intersecting spheres, to 5) two separate spheres.

The purpose of this study is to determine the shell of optimum proportions and configuration from all the shells considered and to evaluate the penalty incurred by using a shell of other than optimum proportions and configuration.

The shell of optimum proportions and configuration is defined as that shell with the lowest ratio of structural weight to shell displacement, given the conditions that all shells are made of the same material and are designed to withstand the same hydrostatic pressure.

A relationship between the weight to displacement ratio (ratio of the structural weight of the shell to the weight of the water that the shell displaces) and the geometric ratios of the shell will be developed. From this relationship it will be possible to determine the effect that purely geometric considerations have upon the weight to displacement ratio.

The weight of reinforcements at the junction of the spherical shells to the cylindrical shells and the weight of stiffening within the cylindrical shells were not included in the weight to displacement ratio. That is the shells were considered to be unstiffened and the intersections unreinforced. Also the shells were considered to be thin shell sections so that the volume of the structural material of the shells can be considered to be the surface area of the shells multiplied by the thickness of the shells.

PROCEDURE

In the design of deep diving vehicles, the hull structural weight to hull displacement ratio is a very important consideration since it governs the amount of weight remaining for propulsion machinery and for payload. Obviously, if less weight is required for the hull structural material, then more weight is available for use as additional machinery or additional payload.

For this reason it was decided to evaluate possible hull configurations and proportions by means of the weight to displacement ratio criterion. The weight to displacement ratio can effectively be divided into two parts. One part is a function of the properties of the material from which the hull is made and a function of the hydrostatic pressure that the hull is designed to withstand. The second part is a function of the geometry of the hull. It is this second part that this investigation will analyze.

Since spherical shells and cylindrical shells are known to give good weight to displacement ratios it was decided to analyze shells consisting of two spherical shell sections joined together by a cylindrical shell section. In the limit, when the length of the cylindrical shell section becomes equal to zero, this becomes a shell consisting of two intersecting spherical shell sections. Further, when the two intersecting spherical shell sections become hemispherical sections of equal radii the shell degenerates to a spherical shell.

There are several possible combinations of geometric ratios which will define the proportions of the shells considered. The three independent

geometric ratio selected for use in this study are: 1) the ratio of the larger spherical shell radius to the smaller spherical shell radius, 2) the ratio of the cylindrical shell radius to the smaller spherical shell radius, and 3) the ratio of the cylindrical shell length to the smaller spherical shell radius. All other possible geometric ratios can be expressed as combinations of the three geometric ratios selected. Other possible geometric ratios which would be useful for design purposes include the length to breadth ratio of the shell.

In the development of the equation relating the weight to displacement ratio to the geometric ratios of the shells, the shells were considered to be unstiffened and the intersections of the shells were considered to be unreinforced. Also, the shells were considered to be thin shell sections made from the same material and designed to withstand the same hydrostatic pressure. The development of the equation relating the weight to displacement ratio to the geometric ratios of the shells is given in Appendix A. This equation is as follows:

$$\frac{W}{D} = \frac{\rho_s}{\rho_w}(6x)\frac{1 + \sqrt{1-c^2} + a^3 + a^2\sqrt{a^2-c^2} + 2c^2f}{2 + 2\sqrt{1-c^2} + c^2\sqrt{1-c^2} + 2a^3 + 2a^2\sqrt{a^2-c^2} + c^2\sqrt{a^2-c^2} + 3c^2f} \quad (1)$$

where:

W is the structural weight of the shell,

D is the weight of water that the shell displaces,

R is the radius of the smaller spherical section,

T is the thickness of the shell of the smaller spherical section,

r is the radius of the larger spherical section,

t is the thickness of the shell of the larger spherical section,

L is the length of the cylindrical section,

d is the radius of the cylindrical section,

h is the thickness of the shell of the cylindrical section,

ρ_s is the density of the material of the shell,

ρ_W is the density of salt water (the displaced medium),

$a = \frac{r}{R}$,

$c = \frac{d}{R}$,

$f = \frac{L}{R}$,

$x = \frac{T}{R} = \frac{t}{r} = \frac{h}{2d}$.

The value of x is directly determined by the hydrostatic pressure that the shell is designed to withstand and by the yield strength and other mechanical properties of the shell material. Extensive research has been undertaken to relate the thickness to radius ratio (x) to the critical pressure for spherical shells. References 1, 2, and 3 recommend the use of the empireacal design formula,

$$p = \frac{0.8E\ (T/R)^2}{\sqrt{1-\nu^2}} \tag{2}$$

where:

p is the critical pressure

ν is Poisson's Ratio

E is Young's Modulus

T is the thickness of the shell

R is the outside radius of the spherical shell,

for the elastic buckling strength of near perfect spherical shells.

In the plastic yield failure region the equation,

$$p = 2\,\sigma\,(T/R) \tag{3}$$

where:

p is the critical pressure

σ is the yield strength of the shell material

T is the thickness of the shell

R is the mean radius of the spherical shell,

applies to near perfect spherical shells.

Equation (1) is applicable when failure is by either the buckling or by the yield failure mode so long as there is no cylindrical section present ($f = \frac{L}{R} = 0$). In this special case $x = \frac{T}{R} = \frac{t}{r} = \frac{p}{2\sigma}$ if failure is by the yield mode. If failure is by the buckling mode, $x = \frac{t}{r} = \frac{T}{R} = \frac{1}{2}\sqrt{\frac{5p\sqrt{1-\nu^2}}{E}}$

When a cylindrical shell section is present, equation (1) is applicable only in the yield mode of failure because the following equation for the collapse pressure of a cylindrical section was used in the development of equation (1),

$$p = \left(\frac{h}{d}\right)\sigma \tag{4}$$

where:

p is the critical pressure

σ is the yield strength of the shell material

h is the shell thickness

d is the radius of the cylindrical shell.

In the general case, $x = \frac{T}{R} = \frac{t}{r} = \frac{h}{2d} = \frac{p}{2\sigma}$ when failure by the yield mode of failure is considered.

Equation (1) was developed assuming failure would be by the yield mode because most practical hulls designed today fall into the yield failure region. Hulls made of materials such as steel, aluminum alloys, and titanium alloys would be extremely thin and would fail at much lower pressures than those of interest, if the spherical sections failed in the buckling mode. Only when materials with very high strength to weight ratios, such as glass, are used in spherical shells does the buckling mode of failure become of practical importance. When this is the case, the weight savings due to the optimum geometric proportions of shells investigated in this paper become of secondary importance.

Equation (4) is valid only when the cylindrical shell is very short or when the shell is stiffened at intervals. The spacing of the intervals where stiffening is required is dependent upon the shell thickness to diameter ratio and upon the mechanical properties of the shell material. Thus, equation (1) would become discontinuous and it would become difficult to determine the general trend in the weight to displacement ratio due to a change in geometry of the shell if the weight of the required stiffeners was considered. Also, the weight of the required stiffeners is of secondary importance to the weight of the cylindrical shell.

Equation (1) can be written as

$$\frac{W}{D} = (M)(G) \tag{1a}$$

where:

$$M = \frac{\rho_s}{\rho_w}(x) \tag{5}$$

$$G = 6\,\frac{1 + \sqrt{1-c^2} + a^3 + a^2\sqrt{a^2-c^2} + 2c^2 f}{2 + 2\sqrt{1-c^2} + c^2\sqrt{1-c^2} + 2a^3 + 2a^2\sqrt{a^2-c^2} + c^2\sqrt{a^2-c^2} + 3c^2 f} \tag{6}$$

Thus, M is a function of the hydrostatic pressure the shell is designed to withstand, the properties of the shell material, and the density of the displaced medium. G is a function of the geometry of the shell.

A range of values for $c = \frac{d}{R}$, $f = \frac{L}{R}$, and $a = \frac{r}{R}$ were assigned and values of G were calculated. The results presented in the following section in graphical form show how this measure of the weight to displacement ratio changes with the geometric ratios.

The shell of optimum proportions and configuration is the shell giving the lowest value of G, according to the criterion selected. The value of G for shells of other than optimum proportions and configuration is a measure of the penalty incurred by using such a shell.

RESULTS

That part of the equation relating the weight to displacement ratio to the geometric ratios of the shells which is a function of the shell geometry is given by equation (6) as,

$$G = 6\ \frac{1 + \sqrt{1-c^2} + a^3 + a^2\sqrt{a^2-c^2} + 2c^2 f}{2 + 2\sqrt{1-c^2} + c^2\sqrt{1-c^2} + 2a^3 + 2a^2\sqrt{a^2-c^2} + c^2\sqrt{a^2-c^2} + 3c^2 f} \qquad (6)$$

The geometric ratios ($a = \frac{r}{R}$, $c = \frac{d}{R}$, and $f = \frac{L}{R}$) were systematically and independently varied and a value of G was calculated for each set of values assigned to the geometric ratios.

The value of $a = \frac{r}{R}$ must be greater than or equal to one. The values assigned to $a = \frac{r}{R}$ in this investigation ranged from one to four in 0.25 increments. The value of $f = \frac{L}{R}$ must be greater than or equal to zero. The values assigned to $f = \frac{L}{R}$ in this investigation ranged from zero to two in 0.25 increments. The value of $c = \frac{d}{R}$ must be greater than or equal to zero but less than or equal to one. Because $c = \frac{d}{R}$ occurs as c^2 in equation (6), and because a clearer display of the variation of G with c is obtained if G is plotted against c^2, the geometric ratio c^2 was varied. The values assigned to c^2 in this investigation ranged from zero to one on 0.1 increments.

Graph 1 shows G as a function of $c^2 = (\frac{d}{R})^2$ for incremental values of $a = \frac{r}{R}$ with the value of $f = \frac{L}{R}$ held constant and equal to zero. Thus, graph 1 shows the variation of weight to displacement ratio for two intersecting spherical shell sections as a function of the ratio of the radius of the intersection circle to the radius of the smaller spherical shell section for incremental values of the ratio of the radii of the spherical shell sections.

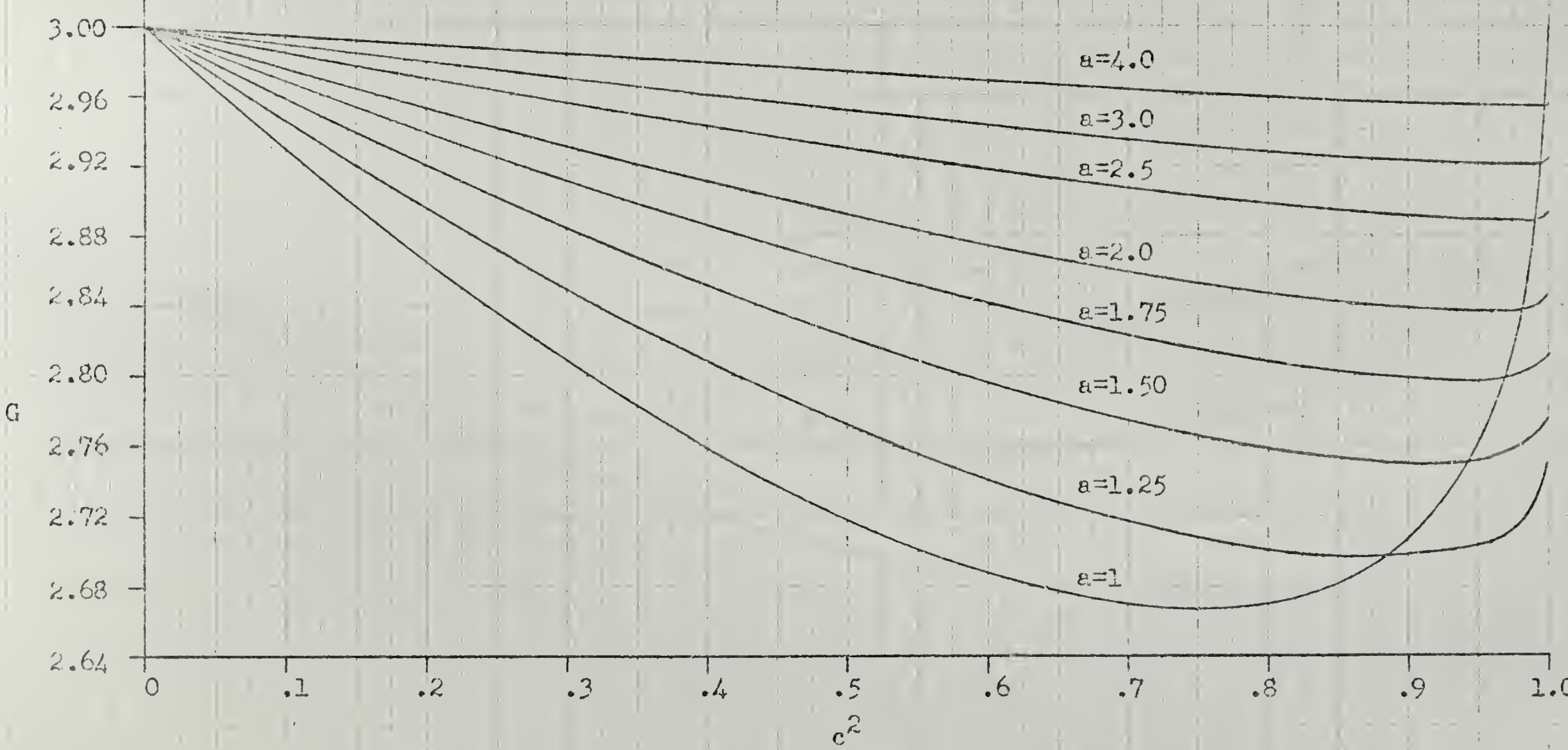
Graphs 2 through 9 show G as a function of $c^2 = (\frac{d}{R})^2$ for incremental values of $f = \frac{L}{R}$ with the value of $a = \frac{r}{R}$ held constant for each graph. Thus, graphs 2 through 9 show the effect of the cylindrical shell section upon the weight to displacement ratio of the shells considered.

The value of $c = \frac{d}{R}$ was chosen as the variable geometric ratio because in the design of hulls, the diameter of the cylindrical shell section often must be large enough to allow access between the two spherical shell sections.

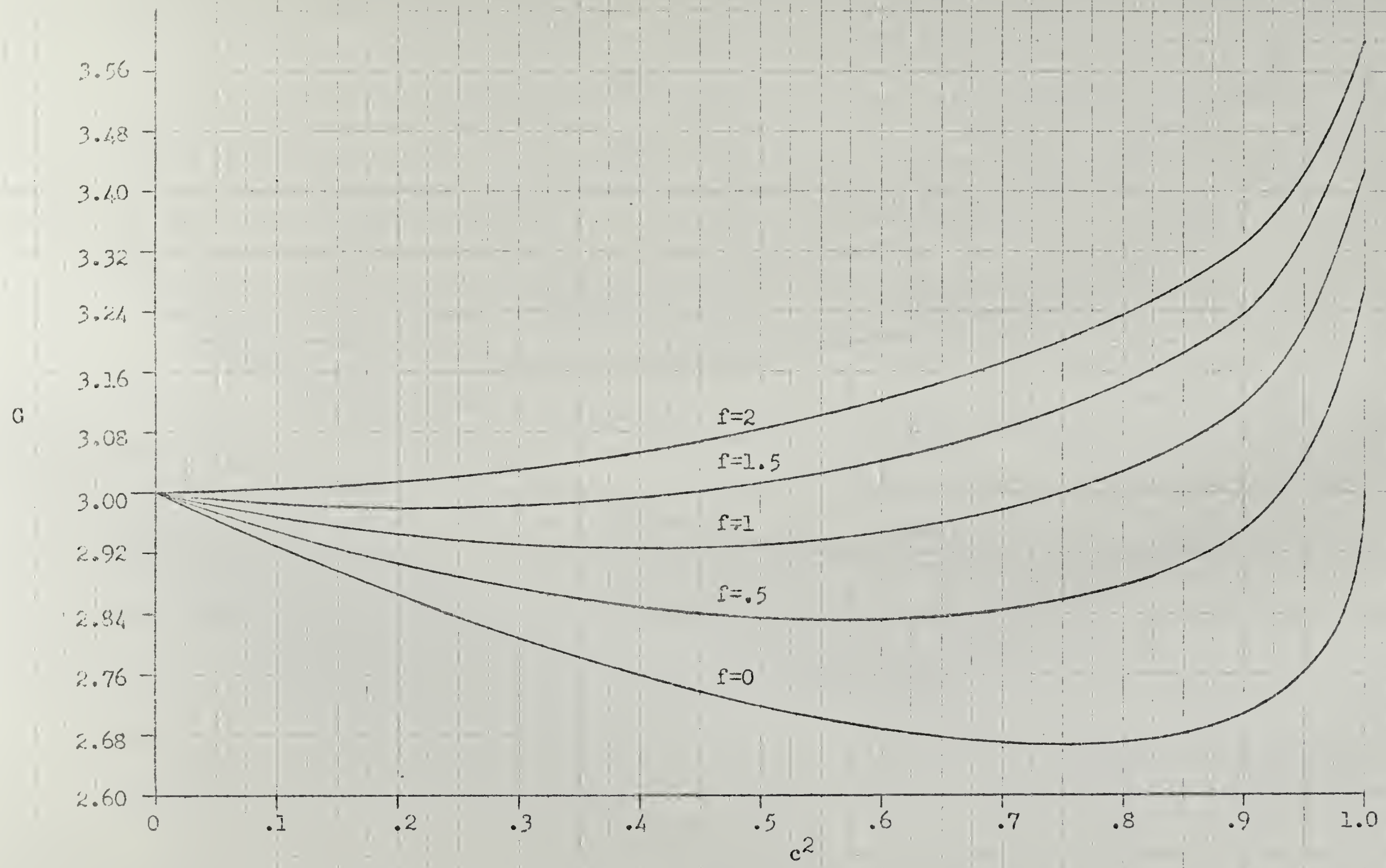
In all cases, a value of $c^2 = (\frac{d}{R})^2 = 0$ represents the case of two separate spherical shells and yields a value of G equal to three so that

$\frac{W}{D} = 3M = 3\frac{\rho_s}{\rho_W} \quad (x) \quad = \quad (\frac{3p}{2\sigma}) \; (\frac{\rho_s}{\rho_W})$ for the case of a spherical shell. When $c = \frac{d}{R} = 1$ the smaller spherical shell section is a hemispherical shell section.

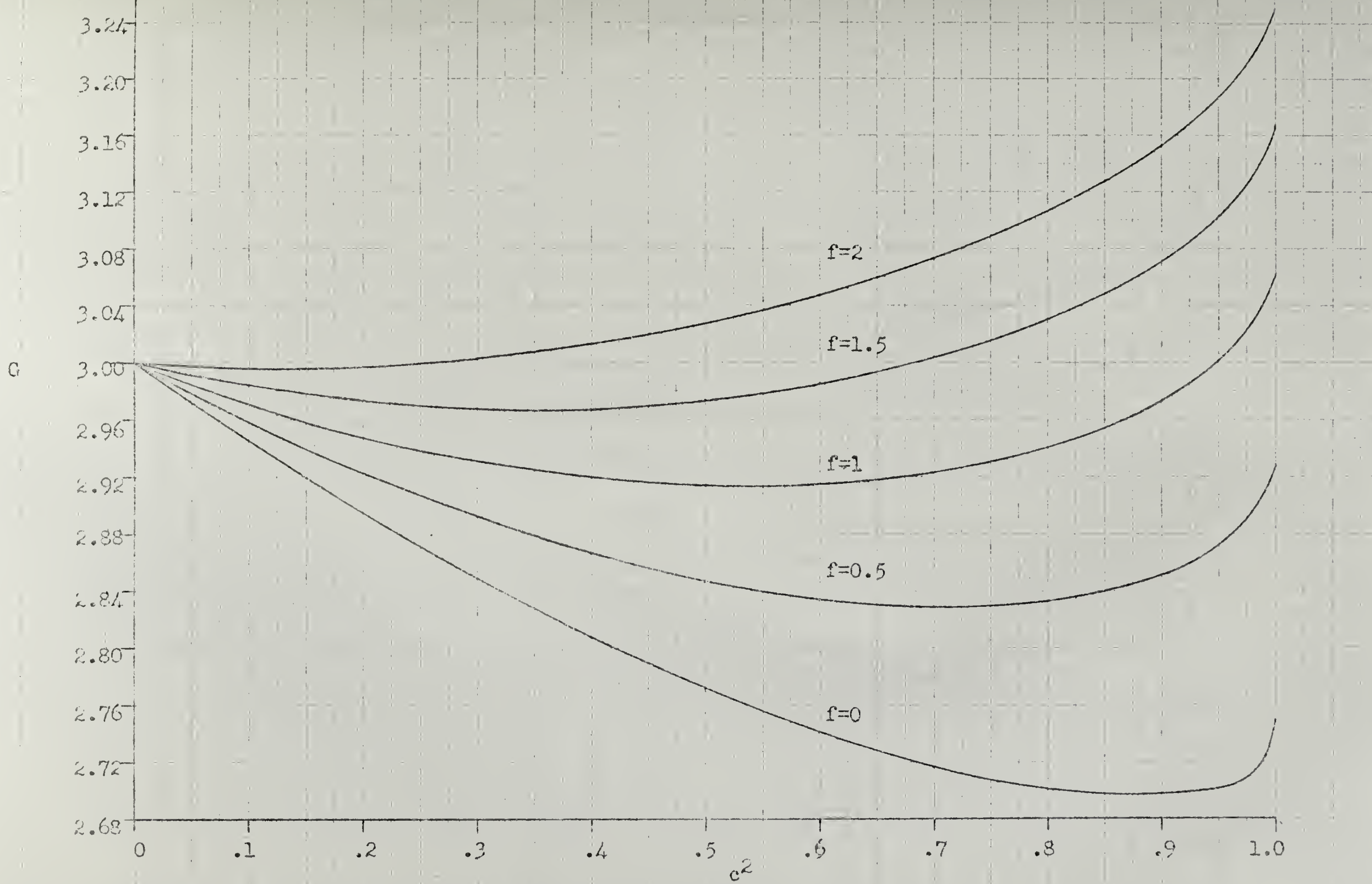
Appendix C is a tabulation of the results obtained by systematically and independently varying the three geometric ratio considered in this investigation through the range of values stated above. The results are presented to allow the reader to plot the results in a different form if he so desires.



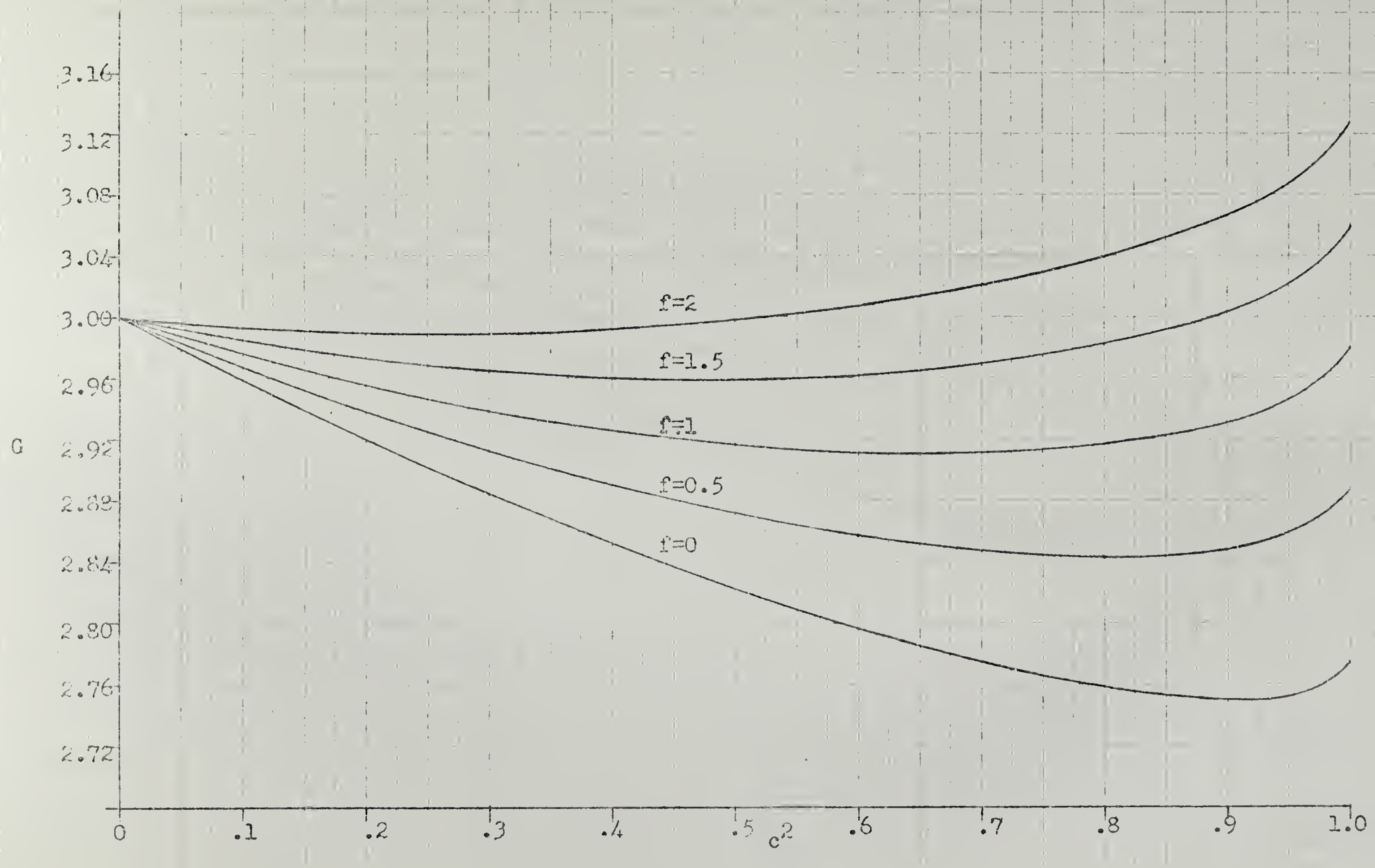
GRAPH 1 --- G vs c^2 for various values of $a = r/R$ with $f = L/R = 0$.



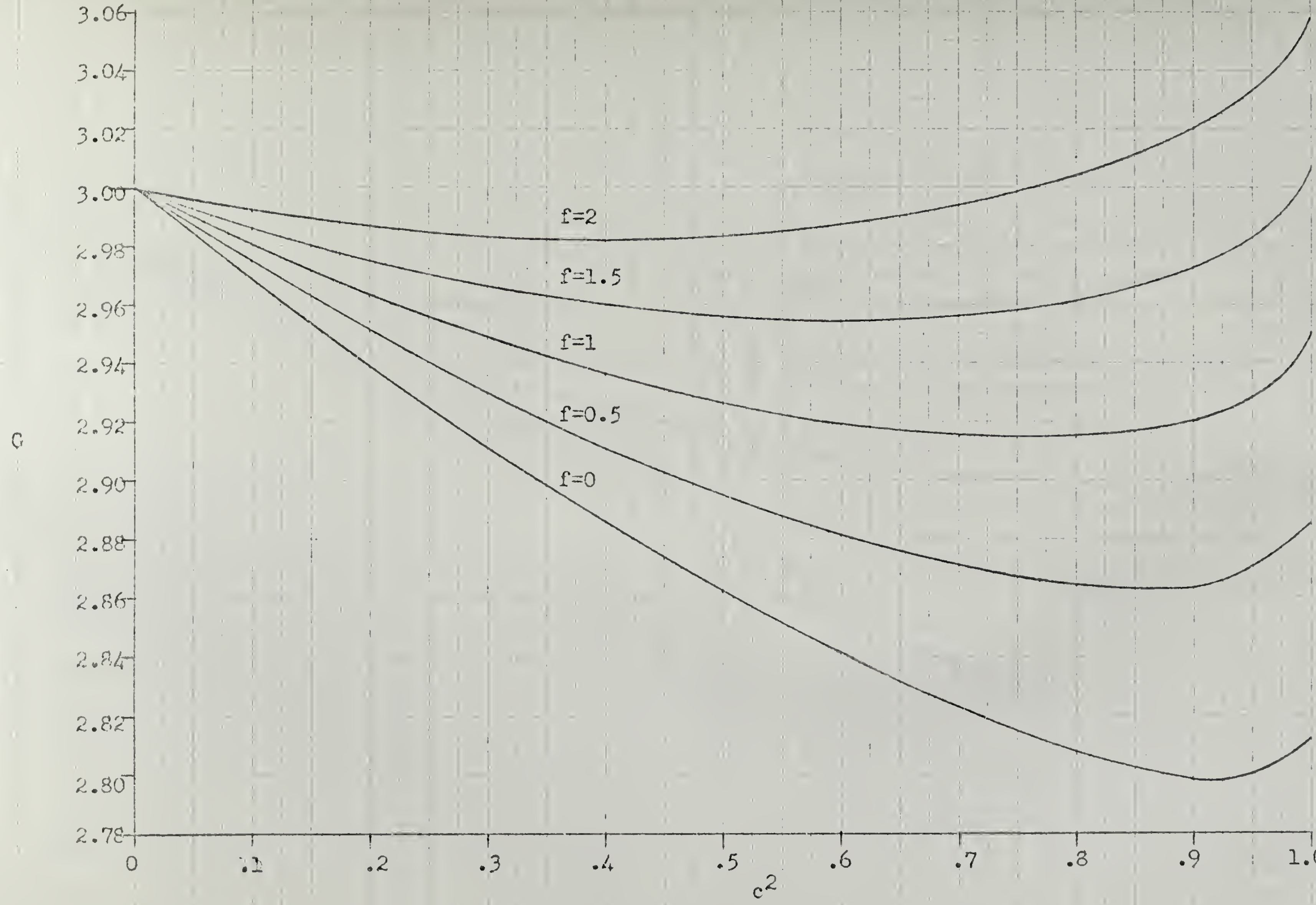
GRAPH 2 --- G vs c^2 for various values of $f = L/R$ with $a = r/R = 1$



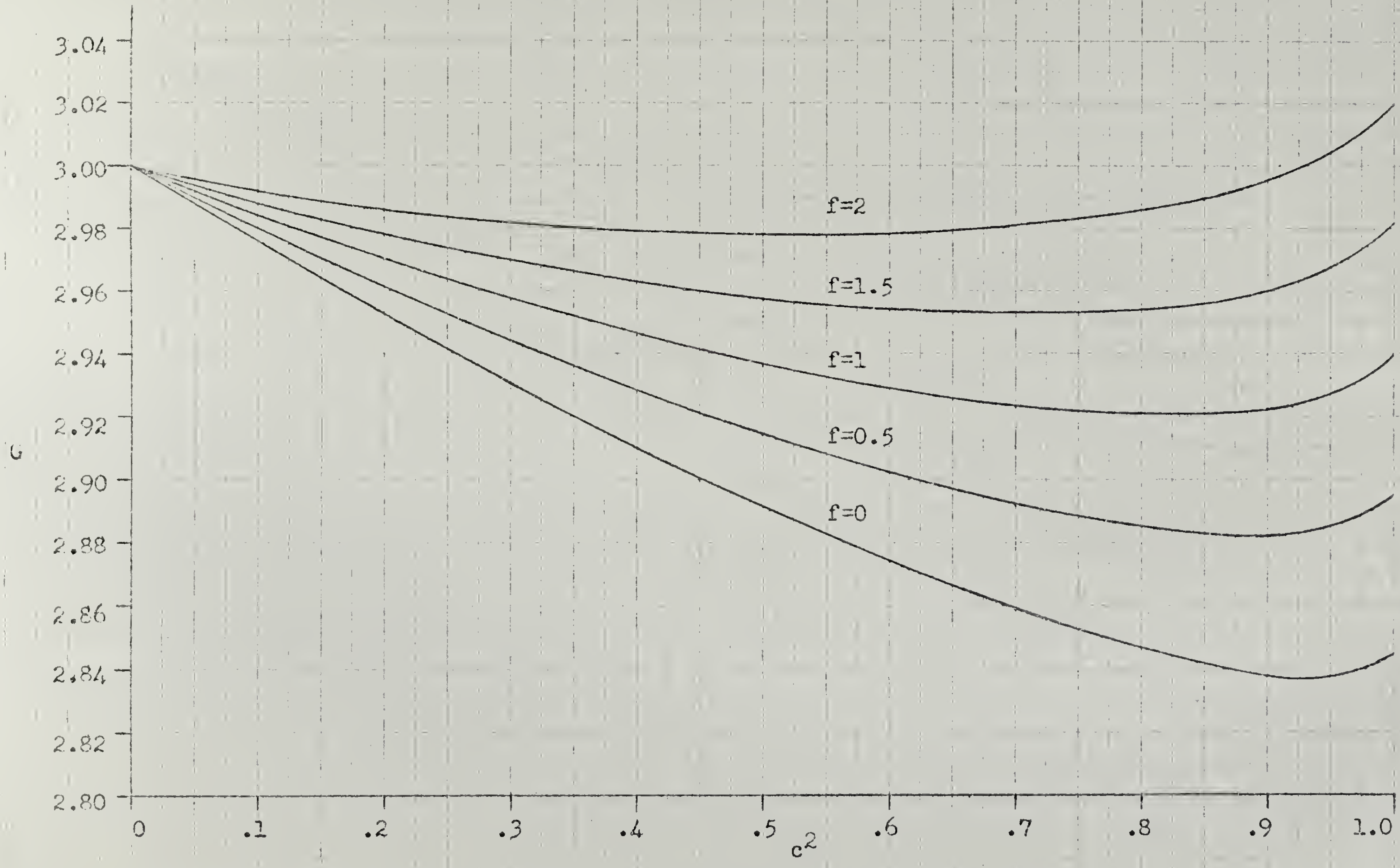
GRAPH 3 --- G vs c^2 for various values of $f = L/R$ with $a = r/R = 1.25$



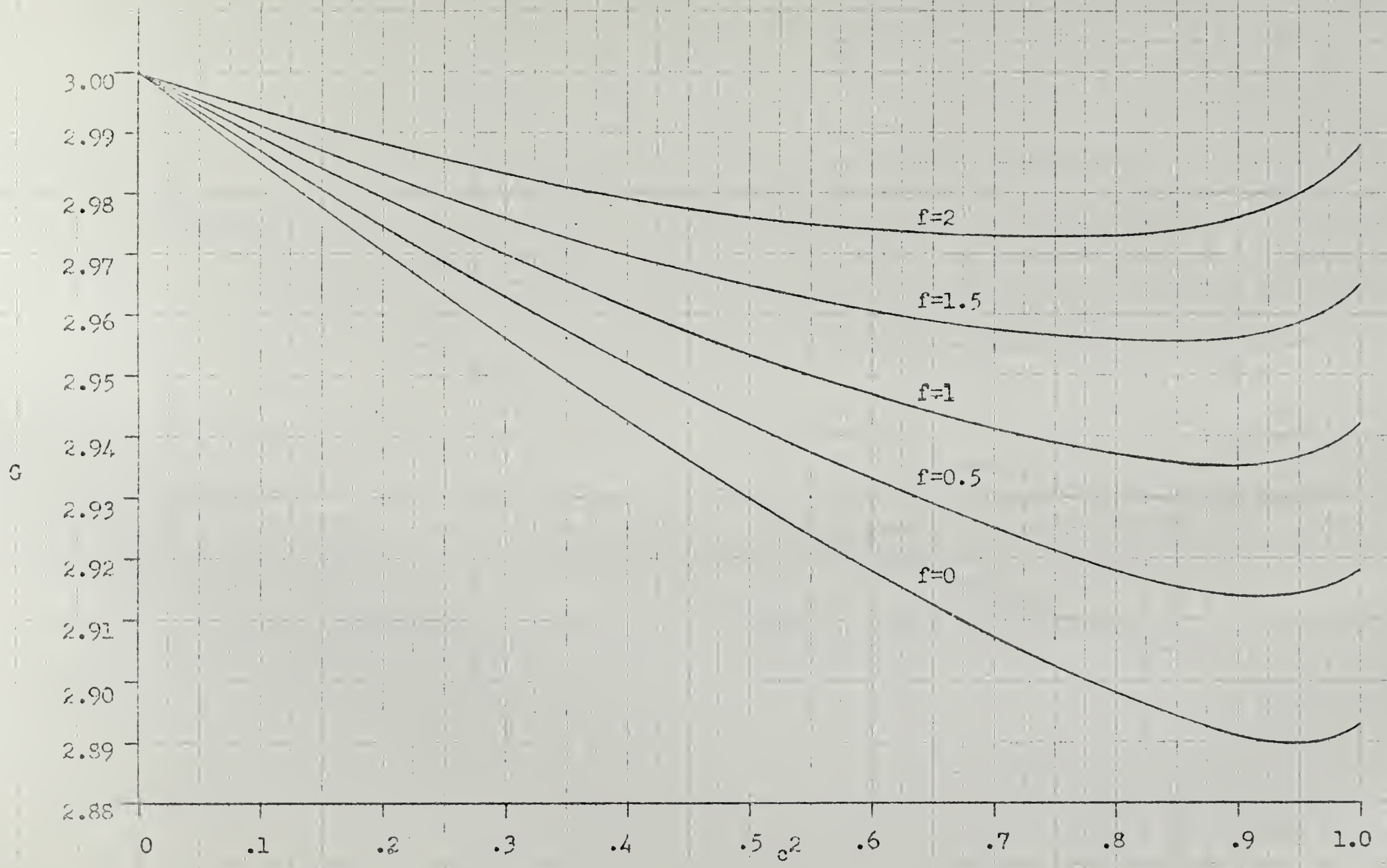
GRAPH 4 --- G vs c^2 for various values of $f = L/R$ with $a = r/R = 1.50$



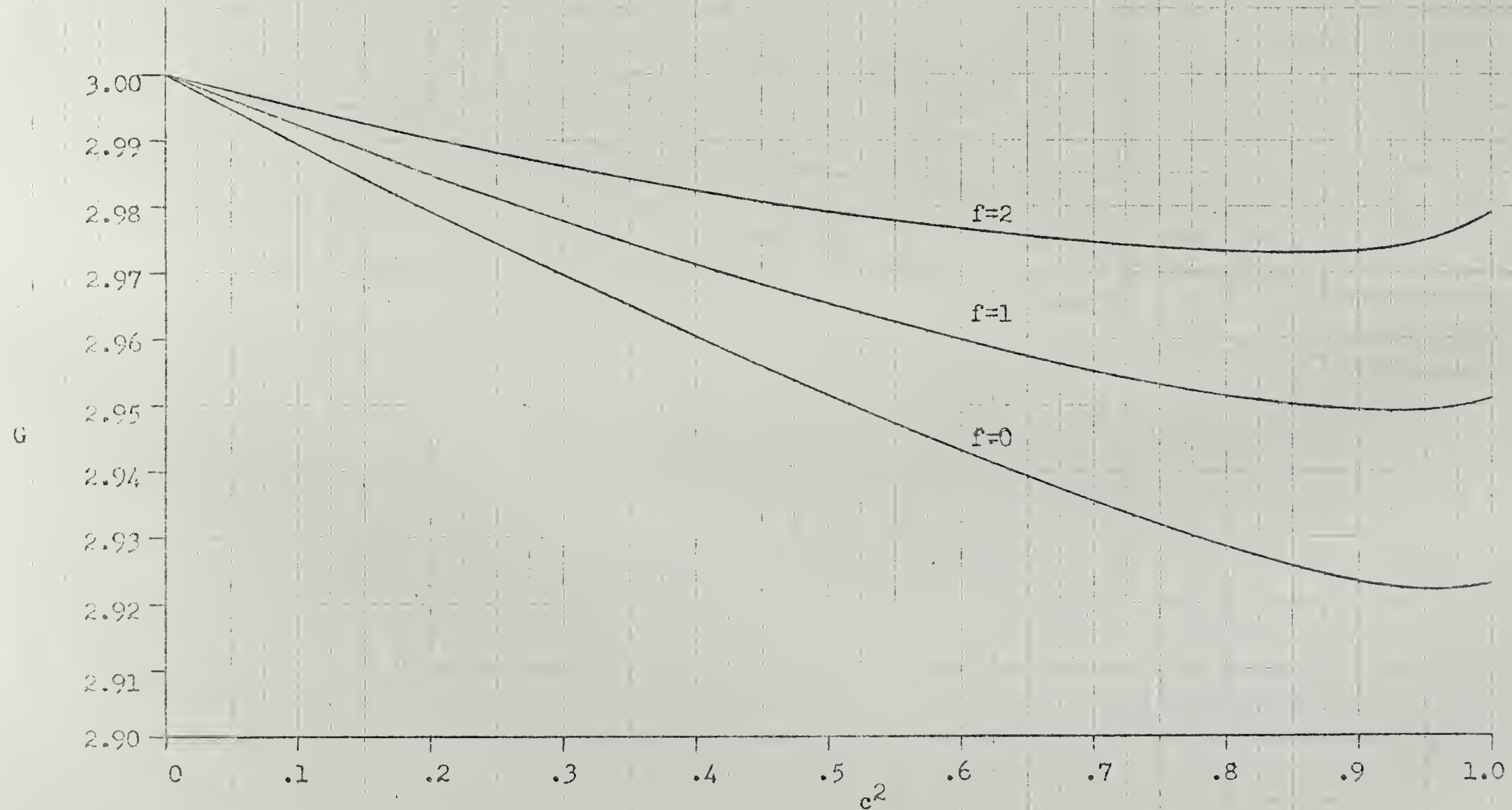
GRAPH 5 --- G vs c^2 for various values of $f = L/R$ with $a = r/R = 1.75$



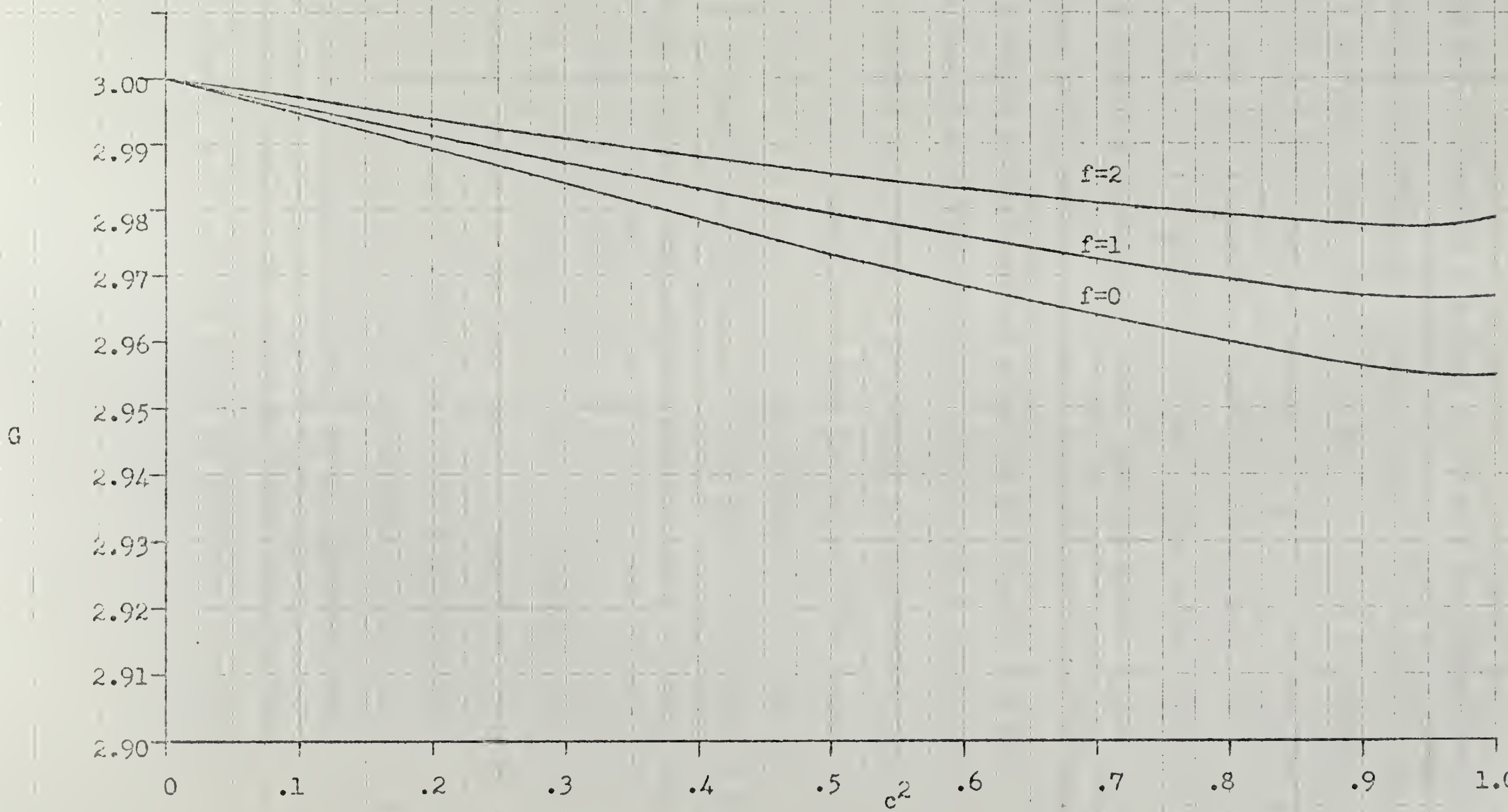
GRAPH 6 --- G vs c^2 for various values of $f = L/R$ with $a = r/R = 2.00$



GRAPH 7 --- G vs c^2 for various values of $f = L/R$ with $a = r/R = 2.50$



GRPAH 8 --- G vs c^2 for various values of f = L/R with a = r/R = 3.00



GRAPH 9 --- G vs c^2 for various values of $f = L/R$ with $a = r/R = 4.00$

DISCUSSION OF RESULTS

1. The results shown in graphs 1 through 9 indicate the minimum weight to displacement ratio for shells of the geometry considered occurs when the radii of the spherical shell section are equal ($a = \frac{r}{R} = 1$) and when there is no cylindrical shell section present ($f = \frac{L}{R} = 0$). An analysis of equation (1) shows that the minimum weight to displacement ratio cannot occur at a value of $f = \frac{L}{R}$ other than zero. Then substituting a value of $a = \frac{r}{R} = 1.01$ into equation (1) shows that the value of the weight displacement ratio for $a = 1.01$ and $f = 0$ is slightly greater than the value of the weight to displacement ratio for $a = 1$ and $f = 0$ for corresponding values of $c^2 = (\frac{d}{R})^2$ between 0 and 3/4. At a values of c^2 slightly greater than 3/4 the value of the weight to displacement ratio for $a = 1.01$ and $f = 0$ is slightly less than the value of the weight to displacement ratio for $a = 1$ and $f = 0$ at corresponding of c^2, but the minimum value of the weight to displacement ratio for $a = 1.01$ and $f = 0$ is slightly greater than the minimum value attained when $a = 1$ and $f = 0$ and occurs at a value of c^2 slightly greater than 3/4.

Calculations shown in Appendix B show that the minimum value of the weight to displacement ratio for the case when $a = 1$ and $f = 0$ is equal to $\frac{8}{3}\left(\frac{\rho_s}{\rho_w}\right)(x)$ and occurs when $c^2 = 3/4$. This gives a pressure hull consisting of two intersecting spherical shell sections with equal radii and with a center to center separation of the spherical shell sections equal to their radii. The length to breadth ratio of this pressure hull is equal to 1.5 and the weight to displacement ratio is 89% of the weight to displacement ratio for a single spherical shell.

2. Pressure hulls consisting of two intersecting shells (with the weight of reinforcement neglected) have a weight to displacement ratio less than the weight to displacement ratio for a pressure hull consisting of a single spherical shell but this improvement in the weight to displacement ratio is small unless the radii of the two spherical shells are nearly the same and the ratio of the radius of the circle of intersection to the radius of the smaller spherical shell is relatively large.

3. Pressure hulls consisting of two spherical shells joined together by a cylindrical shell have weight to displacement ratios which can either be greater than or less than the weight displacement ratio of pressure hulls consisting of a single spherical shell. In general, the weight to displacement ratio increases rapidly as the length of the cylindrical shell section increases.

4. Pressure hulls consisting of two spherical shells of nearly the same radius ($a = \frac{r}{R}$, less than 2) connected together by a cylindrical shell of a relatively short length can be designed so that their weight to displacement ratios are less than the weight to displacement ratio of a pressure hull consisting of a single spherical shell. For the case when the length of the connecting cylinder is zero (two intersecting spherical shells) this is always the case. An advantage of pressure hulls consisting of two spherical shells joined together by a short cylindrical shell (including the case of cylindrical shells of zero length) over pressure hulls consisting of a single spherical shell is that a pressure hull with a length to breadth ratio near to the value of two and a weight to displacement ratio near to that of a hull consisting of a single spherical shell can be obtained.

[illegible]

5. Even though pressure hulls consisting of two spherical shells connected by a cylindrical shell (including the case of two intersecting spherical shells) can be designed with a weight to displacement ratio less than that of a pressure hull consisting of a single spherical shell, this decrease in the weight to displacement ratio is small. Therefore, materials with a higher strength to weight ratio will have to be used for vehicles designed to operate at the deeper reaches of the ocean in order to attain a satisfactory weight to displacement ratio. In other words the value of M must be reduced ($M = p\rho_s/2\sigma \;\; \rho_w$) by increasing the $\frac{\sigma}{\rho_s}$ ratio (strength to weight ratio).

CONCLUSIONS

1. The shell of optimum proportions and configuration was found to be a shell consisting of two intersecting spherical shell sections with equal radii and with a center to center separation of the spherical shell sections equal to their radii. The weight to displacement ratio of this shell was found to be 11.1% less than that of a single spherical shell given that the two shells compared were made of the same material and designed to withstand the same hydrostatic pressure.

2. Even though the weight of required stiffeners was not taken into account in this study, it is expected that a decrease in the weight to displacement ratio can be accomplished when the weight of stiffening is accounted for in the calculations of the weight to displacement ratio.

3. Since stiffening will be required at the intersection of the shells and within the cylindrical shell, the results presented in this investigation should be used only as guides toward a pressure hull design, showing the general trend in the weight to displacement ratio due to a change in the geometry of the pressure hull.

[illegible]

RECOMMENDATIONS

1. Further study should be undertaken to evaluate the importance of the stiffening and reinforcement considerations upon the results obtained in this investigation. The weight of required stiffening and reinforcement was neglected in this investigation. It is realized that the inclusion of the weight of stiffening and reinforcement will increase the weight to displacement ratios obtained but the importance of this increase has not been evaluated.

REFERENCES

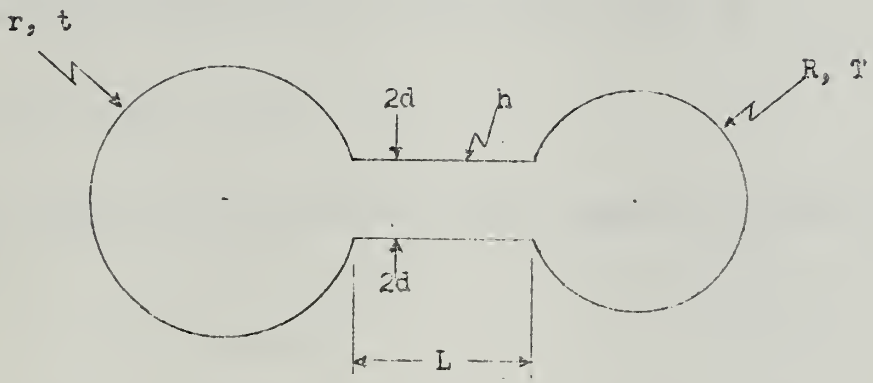
1. Krenzke, M.A., "Tests of Machined Deep Spherical Shells under External Hydrostatic Pressure", David Taylor Model Basin Report 1601 (May 1962).

2. Krenzke, M.A., "The Elastic Buckling Strength of Near-Perfect Deep Spherical Shells with Idean Boundaries", David Taylor Model Basin Report 1713 (July 1963).

3. Krenzke, M.A. and Kierman, T.J., "Tests of Stiffened and Unstiffened Machined Spherical Shells under External Hydrostatic Pressure", David Taylor Model Basin Report 1741 (August 1963).

APPENDIX A

Development of Weight to Displacement Ratio Relationship to the Geometric Ratios



h is the thickness of cylindrical shell

d is the radius of cylindrical shell

L is the length of cylindrical shell

R is the radius of smaller spherical shell

T is the thickness of smaller spherical shell

r is the radius of larger spherical shell

t is the thickness of larger spherical shell

ρ_S is the density of the shell material

ρ_W is the density of salt water (the displaced medium)

π = constant = 3.141592----

The volume displaced by the spherical shells is,

$$V_1 = 2/3\pi R^3 + 2/3\pi r^3 + 1/6\pi\sqrt{R^2-d^2}\left[3R^2 + 3d^2 + (R^2-d^2)\right]$$
$$+ 1/6\pi\sqrt{r^2-d^2}\left[3r^2 + 3d^2 + (r^2-d^2)\right] \tag{1}$$

Assuming higher order terms of t/r and T/R can be neglected, the volume of the spherical shells is,

$$V_2 = 2\pi R^2T + 2\pi RT\sqrt{R^2-d^2} + 2\pi r^2t + 2\pi rt\sqrt{r^2-d^2} \tag{2}$$

The volume displaced by the cylindrical shell is,

$$V_3 = \pi d^2L \tag{3}$$

The volume of the cylindrical shell, neglecting higher order terms of h/d is,

$$V_4 = 2\pi dhL \tag{4}$$

The weight-displacement ratio can now be written as,

$$\frac{W}{D} = \frac{\rho_s}{\rho_w} \frac{V_2 + V_4}{V_1 + V_3} \tag{5}$$

providing all shells are made of materials of the same density.

If both the spherical shells and the cylindrical shell are to be designed to fail by the yield mode of failure then,

$$p = 2\sigma_y\frac{T}{R} = 2\sigma_y\frac{t}{r} \qquad \text{(spherical shells)} \tag{6}$$

$$p = 2\sigma_y\left(\frac{h}{2d}\right) \qquad \text{(cylindrical shell)} \tag{7}$$

so if the spherical shells and cylindrical shells are designed to the same pressure then,

$$X = \frac{T}{R} = \frac{t}{R} = \frac{h}{2d} \tag{8}$$

This assumes the cylindrical shell is stiffened so that the equation $p = 2\sigma y(\frac{h}{2d})$ applies, but in the development for the volume displaced by the cylindrical shell, the volume (and thus the weight) of the stiffeners was neglected.

Substituting equations 1, 2, 3, and 4 into equation 5 one has

$$\frac{W}{D} = \frac{\frac{\rho_s}{\rho_w}\left[2(\frac{T}{R}) + 2(\frac{T}{R})\sqrt{1-(\frac{d}{R})^2} + 2(\frac{t}{r})(\frac{r}{R})^3 + 2(\frac{t}{r})(\frac{r}{R})^2\sqrt{(\frac{r}{R})^2-(\frac{d}{R})^2} + 4(\frac{d}{R})^2\,(\frac{L}{R})(\frac{h}{2d})\right]}{\frac{2}{3} + \frac{2}{3}\sqrt{1-(\frac{d}{R})^2} + \frac{1}{3}(\frac{d}{R})^2\sqrt{1-(\frac{d}{R})^2} + \frac{2}{3}(\frac{r}{R})^3 + \frac{2}{3}(\frac{r}{R})^2\sqrt{(\frac{r}{R})^2-(\frac{d}{R})^2} + \frac{1}{3}(\frac{d}{R})^2\sqrt{(\frac{r}{R})^2-(\frac{d}{R})^2} + (\frac{d}{R})^2(\frac{L}{R})} \quad (9)$$

using equation 8,

$$\frac{W}{D} = (\frac{\rho_s}{\rho_w})\,(6x)\,\frac{1 + \sqrt{1-c^2} + a^3 + a^2\sqrt{a^2-c^2} + 2c^2 f}{2 + 2\sqrt{1-c^2} + c^2\sqrt{1-c^2} + 2a^3 + 2a^2\sqrt{a^2-c^2} + c^2\sqrt{a^2-c^2} + 3c^2 f} \quad (10)$$

where:

a is equal to r/R

c is equal to d/R

f is equal to L/R.

Analyzing this equation in special cases one can bring out several pertinent points.

Special Case I: $c = 0$

This gives the case where one has two separate spheres. $\frac{W}{D} = (3x)\,\frac{\rho_s}{\rho_w}$ for this special case.

Special Case II: $c = 1$, $a = 1$, and $f = 0$

This gives the case of a single sphere, Again, $\frac{W}{D} = (3x)\,\frac{\rho_s}{\rho_w}$.

APPENDIX B

Analytical Solution for Minimum Weight-Displacement Ratio for the Special Case $a = \frac{r}{R} = 1$ and $f = \frac{L}{R} = 0$ (spherical shells of equal radius, and cylindrical shell of a length of zero).

For this case,

$$\frac{W}{D} = (6x)\left(\frac{\rho_s}{\rho_w}\right)\frac{1 + \sqrt{1-c^2} + 1 + \sqrt{1-c^2}}{2 + 2\sqrt{1-c^2} + c^2\sqrt{1-c^2} + 2 + 2\sqrt{1-c^2} + c^2\sqrt{1-c^2}} \qquad (1)$$

Simplifying equation 1,

$$\frac{W}{D} = (6x)\left(\frac{\rho_s}{\rho_w}\right)\frac{1 + \sqrt{1-c^2}}{2 + 2\sqrt{1-c^2} + c^2\sqrt{1-c^2}} \qquad (2)$$

$\frac{\partial}{\partial c}\left(\frac{W}{D}\right) = 0$; yields,

$$(2 + 2\sqrt{1-c^2} + c^2\sqrt{1-c^2})\left(-\frac{c}{\sqrt{1-c^2}}\right) = (1 + \sqrt{1-c^2})\left(-\frac{2c}{\sqrt{1-c^2}} + \frac{c^3}{\sqrt{1-c^2}} - 2c\sqrt{1-c^2}\right)$$

$$2c + 2c\sqrt{1-c^2} + c^3\sqrt{1-c^2} = 3c^3 + 3c^3\sqrt{1-c^2}$$

$$c\ [3c^2 + 2c^2\sqrt{1-c^2} - 2\sqrt{1-c^2} - 2] = 0$$

$$\frac{3c^2 - 2}{\sqrt{1-c^2}} = 2 - 2c^2; \quad c = 0$$

$$9c^4 - 12c^2 + 4 = 4 - 12c^2 + 12c^4 - 4c^6; \quad c = 0$$

$$c = 0;\ c^4 = 0;\ 4c^2 - 3 = 0$$

$c^2 = 3/4$ is a minimum

$$\frac{W}{D}\, c^2 = 3/4 = (6x)\left(\frac{\rho_s}{\rho_w}\right)\frac{1 + \frac{1}{2}}{2 + 1 + 3/8} = (6x)\left(\frac{\rho_s}{\rho_w}\right)\frac{12}{27} = \frac{8}{3}\left(\frac{\rho_s}{\rho_w}\right)(x)$$

total length $= 2R + 2\ [R\sqrt{1-c^2}]$

total length $(c^2 = 3/4) = 3R$

APPENDIX C

Tabulation of the Results Obtained by Varying the Geometric Ratios

The equation relating the weight to displacement ratio to the geometric ratios of the shells considered in this investigation (equation 1) can be written as,

$$\frac{W}{D} = (M)(G)$$

where $M = (x)\ (\frac{\rho_s}{\rho_w})$

$$G = (6)\ \frac{1 + \sqrt{1-c^2} + a^3 + a^2\ \sqrt{a^2-c^2} + 2c^2f}{2 + 2\ \sqrt{1-c^2} + c^2\ \sqrt{1-c^2} + 2a^3 + 2a^2\ \sqrt{a^2-c^2} + c^2\ \sqrt{a^2-c^2} + 3c^2f}$$

The following is a tabulation of values of G calculated for variations in $a = r/R$, $c^2 = (\frac{d}{R})^2$ and $f = L/R$.

a=1	f=0.00	f=0.25	f=0.50	f=0.75	f=1.00
c^2	G	G	G	G	G
0.0	3.000	3.000	3.000	3.000	3.000
0.1	2.929	2.939	2.948	2.958	2.968
0.2	2.865	2.886	2.906	2.926	2.945
0.3	2.808	2.841	2.873	2.903	2.931
0.4	2.759	2.806	2.849	2.889	2.926
0.5	2.718	2.779	2.834	2.885	2.931
0.6	2.688	2.764	2.832	2.893	2.948
0.7	2.669	2.763	2.844	2.915	2.978
0.8	2.670	2.782	2.877	2.958	3.028
0.9	2.707	2.841	2.950	3.040	3.116
1.0	3.000	3.157	3.272	3.359	3.423

a=1	f=1.25	f=1.50	f=1.75	f=2.00
c^2	G	G	G	G
0.0	3.000	3.000	3.000	3.000
0.1	2.977	2.986	2.995	3.004
0.2	2.963	2.980	2.997	3.014
0.3	2.958	2.983	3.007	3.030
0.4	2.961	2.994	3.025	3.053
0.5	2.974	3.013	3.050	3.083
0.6	2.997	3.042	3.084	3.122
0.7	3.034	3.084	3.129	3.170
0.8	3.090	3.144	3.192	3.235
0.9	3.181	3.237	3.286	3.329
1.0	3.483	3.529	3.567	3.599

a=1.25	f=0.00	f=0.25	f=0.50	f=0.75	f=1.00
c^2	G	G	G	G	G
0.0	3.000	3.000	3.000	3.000	3.000
0.1	2.945	2.952	2.958	2.965	2.971
0.2	2.895	2.909	2.922	2.936	2.948
0.3	2.849	2.871	2.892	2.912	2.931
0.4	2.808	2.838	2.867	2.894	2.920
0.5	2.772	2.811	2.847	2.882	2.914
0.6	2.741	2.789	2.834	2.876	2.915
0.7	2.717	2.775	2.829	2.878	2.923
0.8	2.701	2.770	2.833	2.889	2.941
0.9	2.698	2.779	2.851	2.915	2.973
1.0	2.750	2.846	2.928	3.000	3.062

a=1.25	f=1.25	f=1.50	f=1.75	f=2.00
c^2	G	G	G	G
0.0	3.000	3.000	3.000	3.000
0.1	2.978	2.984	2.990	2.996
0.2	2.961	2.973	2.985	2.997
0.3	2.950	2.968	2.986	3.003
0.4	2.945	2.968	2.991	3.013
0.5	2.945	2.974	3.001	3.027
0.6	2.951	2.985	3.017	3.047
0.7	2.965	3.004	3.039	3.073
0.8	2.988	3.030	3.070	3.106
0.9	3.024	3.071	3.113	3.152
1.0	3.117	3.167	3.210	3.250

a=1.50	f=0.00	f=0.25	f=0.50	f=0.75	f=1.00
c^2	G	G	G	G	G
0.0	3.000	3.000	3.000	3.000	3.000
0.1	2.959	2.963	2.967	2.972	2.976
0.2	2.920	2.929	2.938	2.947	2.956
0.3	2.884	2.899	2.913	2.926	2.939
0.4	2.852	2.871	2.890	2.909	2.926
0.5	2.822	2.847	2.872	2.895	2.917
0.6	2.796	2.828	2.857	2.885	2.912
0.7	2.775	2.812	2.847	2.880	2.911
0.8	2.758	2.802	2.843	2.881	2.917
0.9	2.750	2.800	2.847	2.891	2.931
1.0	2.775	2.833	2.887	2.936	2.980

a=1.50	f=1.25	f=1.50	f=1.75	f=2.00
c^2	G	G	G	G
0.0	3.000	3.000	3.000	3.000
0.1	2.980	2.985	2.989	2.993
0.2	2.965	2.973	2.981	2.990
0.3	2.952	2.965	2.977	2.989
0.4	2.943	2.960	2.976	2.992
0.5	2.938	2.959	2.979	2.998
0.6	2.937	2.962	2.985	3.007
0.7	2.941	2.969	2.996	3.021
0.8	2.950	2.982	3.011	3.039
0.9	2.968	3.003	3.036	3.066
1.0	3.021	3.059	3.094	3.127

a=1.75 c^2	f=0.00 G	f=0.25 G	f=0.50 G	f=0.75 G	f=1.00 G
0.0	3.000	3.000	3.000	3.000	3.000
0.1	2.969	2.972	2.975	2.978	2.981
0.2	2.939	2.945	2.951	2.957	2.963
0.3	2.911	2.921	2.930	2.939	2.949
0.4	2.886	2.899	2.911	2.924	2.936
0.5	2.862	2.879	2.895	2.911	2.926
0.6	2.841	2.862	2.881	2.901	2.919
0.7	2.823	2.847	2.871	2.893	2.915
0.8	2.808	2.837	2.864	2.890	2.915
0.9	2.799	2.832	2.863	2.892	2.920
1.0	2.812	2.850	2.885	2.918	2.950

a=1.75 c^2	f=1.25 G	f=1.50 G	f=1.75 G	f=2.00 G
0.0	3.000	3.000	3.000	3.000
0.1	2.983	2.986	2.989	2.992
0.2	2.969	2.975	2.981	2.987
0.3	2.958	2.966	2.975	2.983
0.4	2.948	2.960	2.971	2.982
0.5	2.941	2.956	2.970	2.984
0.6	2.937	2.954	2.971	2.987
0.7	2.936	2.956	2.975	2.994
0.8	2.939	2.961	2.983	3.004
0.9	2.947	2.972	2.997	3.020
1.0	2.979	3.007	3.033	3.058

a=2.00	f=0.00	f=0.25	f=0.50	f=0.75	f=1.00
c^2	G	G	G	G	G
0.0	3.000	3.000	3.000	3.000	3.000
0.1	2.976	2.978	2.980	2.982	2.984
0.2	2.953	2.957	2.961	2.965	2.970
0.3	2.931	2.937	2.944	2.951	2.957
0.4	2.910	2.919	2.928	2.937	2.946
0.5	2.892	2.903	2.914	2.925	2.936
0.6	2.874	2.889	2.902	2.916	2.929
0.7	2.859	2.876	2.892	2.908	2.923
0.8	2.847	2.866	2.885	2.903	2.921
0.9	2.838	2.860	2.882	2.902	2.922
1.0	2.845	2.870	2.895	2.918	2.940

a=2.00	f=1.25	f=1.50	f=1.75	f=2.00
c^2	G	G	G	G
0.0	3.000	3.000	3.000	3.000
0.1	2.986	2.988	2.990	2.992
0.2	2.974	2.978	2.982	2.986
0.3	2.963	2.970	2.976	2.982
0.4	2.954	2.963	2.971	2.979
0.5	2.947	2.957	2.968	2.978
0.6	2.942	2.954	2.966	2.978
0.7	2.938	2.953	2.967	2.981
0.8	2.938	2.954	2.970	2.986
0.9	2.941	2.960	2.978	2.995
1.0	2.961	2.982	3.001	3.020

a=2.25	f=0.00	f=0.25	f=0.50	f=0.75	f=1.00
c^2	G	G	G	G	G
0.0	3.000	3.000	3.000	3.000	3.000
0.1	2.981	2.982	2.984	2.985	2.987
0.2	2.962	2.966	2.969	2.972	2.975
0.3	2.945	2.950	2.955	2.959	2.964
0.4	2.929	2.935	2.942	2.948	2.954
0.5	2.913	2.922	2.930	2.938	2.946
0.6	2.899	2.909	2.919	2.929	2.938
0.7	2.887	2.899	2.910	2.921	2.933
0.8	2.876	2.890	2.903	2.916	2.929
0.9	2.868	2.884	2.899	2.914	2.928
1.0	2.872	2.890	2.907	2.923	2.939

a=2.25	f=1.25	f=1.50	f=1.75	f=2.00
c^2	G	G	G	G
0.0	3.000	3.000	3.000	3.000
0.1	2.988	2.990	2.991	2.993
0.2	2.978	2.981	2.984	2.987
0.3	2.969	2.973	2.978	2.982
0.4	2.960	2.966	2.972	2.978
0.5	2.953	2.961	2.968	2.976
0.6	2.948	2.957	2.966	2.975
0.7	2.943	2.954	2.965	2.975
0.8	2.941	2.953	2.965	2.977
0.9	2.942	2.956	2.969	2.982
1.0	2.955	2.970	2.985	2.999

a=2.50	f=0.00	f=0.25	f=0.50	f=0.75	f=1.00
c^2	G	G	G	G	G
0.0	3.000	3.000	3.000	3.000	3.000
0.1	2.985	2.986	2.987	2.988	2.989
0.2	2.970	2.972	2.974	2.977	2.979
0.3	2.956	2.959	2.963	2.966	2.970
0.4	2.942	2.947	2.952	2.956	2.961
0.5	2.930	2.936	2.942	2.948	2.953
0.6	2.918	2.925	2.933	2.940	2.947
0.7	2.907	2.916	2.925	2.933	2.941
0.8	2.898	2.908	2.918	2.928	2.937
0.9	2.891	2.902	2.914	2.924	2.935
1.0	2.893	2.906	2.918	2.930	2.942

a=2.50	f=1.25	f=1.50	f=1.75	f=2.00
c^2	G	G	G	G
0.0	3.000	3.000	3.000	3.000
0.1	2.990	2.991	2.992	2.994
0.2	2.981	2.983	2.986	2.988
0.3	2.973	2.976	2.980	2.983
0.4	2.966	2.970	2.975	2.979
0.5	2.959	2.965	2.970	2.976
0.6	2.954	2.960	2.967	2.974
0.7	2.949	2.957	2.965	2.973
0.8	2.946	2.956	2.964	2.973
0.9	2.946	2.956	2.966	2.976
1.0	2.954	2.965	2.977	2.988

a=2.75	f=0.00	f=0.25	f=0.50	f=0.75	f=1.00
c^2	G	G	G	G	G
0.0	3.000	3.000	3.000	3.000	3.000
0.1	2.987	2.988	2.989	2.990	2.991
0.2	2.975	2.977	2.979	2.980	2.982
0.3	2.964	2.966	2.969	2.971	2.974
0.4	2.952	2.956	2.960	2.963	2.967
0.5	2.942	2.946	2.951	2.955	2.960
0.6	2.932	2.938	2.943	2.948	2.954
0.7	2.923	2.930	2.936	2.942	2.949
0.8	2.915	2.923	2.930	2.937	2.945
0.9	2.909	2.917	2.926	2.934	2.942
1.0	2.909	2.919	2.928	2.938	2.947

a=2.75	f=1.25	f=1.50	f=1.75	f=2.00
c^2	G	G	G	G
0.0	3.000	3.000	3.000	3.000
0.1	2.992	2.993	2.993	2.994
0.2	2.984	2.986	2.987	2.989
0.3	2.977	2.979	2.982	2.984
0.4	2.970	2.974	2.977	2.981
0.5	2.964	2.969	2.973	2.977
0.6	2.959	2.964	2.970	2.975
0.7	2.955	2.961	2.967	2.973
0.8	2.952	2.959	2.966	2.973
0.9	2.950	2.958	2.966	2.973
1.0	2.956	2.964	2.973	2.981

a=3.00	f=0.00	f=0.25	f=0.50	f=0.75	f=1.00
c^2	G	G	G	G	G
0.0	3.000	3.000	3.000	3.000	3.000
0.1	2.989	2.990	2.991	2.991	2.992
0.2	2.979	2.981	2.982	2.983	2.985
0.3	2.970	2.972	2.974	2.976	2.978
0.4	2.960	2.963	2.966	2.969	2.971
0.5	2.951	2.955	2.958	2.962	2.965
0.6	2.943	2.947	2.951	2.956	2.960
0.7	2.935	2.940	2.945	2.950	2.955
0.8	2.928	2.934	2.940	2.946	2.951
0.9	2.923	2.929	2.936	2.942	2.949
1.0	2.923	2.930	2.937	2.944	2.951

a=3.00	f=1.25	f=1.50	f=1.75	f=2.00
c^2	G	G	G	G
0.0	3.000	3.000	3.000	3.000
0.1	2.993	2.994	2.994	2.995
0.2	2.986	2.987	2.989	2.990
0.3	2.980	2.982	2.984	2.986
0.4	2.974	2.977	2.979	2.982
0.5	2.969	2.972	2.975	2.979
0.6	2.964	2.968	2.972	2.976
0.7	2.960	2.965	2.969	2.974
0.8	2.957	2.962	2.968	2.973
0.9	2.955	2.961	2.967	2.973
1.0	2.958	2.965	2.972	2.979

a=3.25	f=0.00	f=0.25	f=0.50	f=0.75	f=1.00
c^2	G	G	G	G	G
0.0	3.000	3.000	3.000	3.000	3.000
0.1	2.991	2.992	2.992	2.993	2.993
0.2	2.983	2.984	2.985	2.986	2.987
0.3	2.974	2.976	2.978	2.979	2.981
0.4	2.966	2.969	2.971	2.973	2.975
0.5	2.959	2.961	2.964	2.967	2.970
0.6	2.952	2.955	2.958	2.962	2.965
0.7	2.945	2.949	2.953	2.957	2.960
0.8	2.939	2.943	2.948	2.952	2.957
0.9	2.934	2.939	2.944	2.949	2.954
1.0	2.933	2.939	2.945	2.950	2.956

a=3.25	f=1.25	f=1.50	f=1.75	f=2.00
c^2	G	G	G	G
0.0	3.000	3.000	3.000	3.000
0.1	2.994	2.994	2.995	2.995
0.2	2.988	2.989	2.990	2.991
0.3	2.932	2.984	2.986	2.987
0.4	2.977	2.979	2.981	2.934
0.5	2.972	2.975	2.978	2.980
0.6	2.968	2.971	2.975	2.978
0.7	2.964	2.968	2.972	2.976
0.8	2.961	2.965	2.970	2.974
0.9	2.959	2.964	2.969	2.974
1.0	2.961	2.967	2.972	2.977

a=3.50	f=0.00	f=0.25	f=0.50	f=0.75	f=1.00
c^2	G	G	G	G	G
0.0	3.000	3.000	3.000	3.000	3.000
0.1	2.992	2.993	2.993	2.994	2.994
0.2	2.985	2.986	2.987	2.988	2.989
0.3	2.978	2.979	2.981	2.982	2.983
0.4	2.971	2.973	2.975	2.976	2.978
0.5	2.965	2.967	2.969	2.971	2.973
0.6	2.958	2.961	2.964	2.966	2.969
0.7	2.953	2.956	2.959	2.962	2.965
0.8	2.947	2.951	2.955	2.958	2.962
0.9	2.943	2.947	2.951	2.955	2.959
1.0	2.942	2.946	2.951	2.955	2.960

a=3.50	f=1.25	f=1.50	f=1.75	f=2.00
c^2	G	G	G	G
0.0	3.000	3.000	3.000	3.000
0.1	2.995	2.995	2.995	2.996
0.2	2.989	2.990	2.991	2.992
0.3	2.984	2.986	2.987	2.988
0.4	2.980	2.982	2.983	2.985
0.5	2.976	2.978	2.980	2.982
0.6	2.972	2.974	2.977	2.979
0.7	2.968	2.971	2.974	2.977
0.8	2.965	2.969	2.972	2.976
0.9	2.963	2.967	2.971	2.975
1.0	2.964	2.969	2.973	2.977

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

a=3.75	f=0.00	f=0.25	f=0.50	f=0.75	f=1.00
c^2	G	G	G	G	G
0.0	3.000	3.000	3.000	3.000	3.000
0.1	2.993	2.994	2.994	2.995	2.995
0.2	2.987	2.988	2.989	2.989	2.990
0.3	2.981	2.982	2.983	2.984	2.985
0.4	2.975	2.976	2.978	2.979	2.981
0.5	2.969	2.971	2.973	2.975	2.976
0.6	2.964	2.966	2.968	2.970	2.973
0.7	2.959	2.961	2.964	2.966	2.969
0.8	2.954	2.957	2.960	2.963	2.966
0.9	2.950	2.953	2.957	2.960	2.963
1.0	2.949	2.953	2.956	2.960	2.964

a=3.75	f=1.25	f=1.50	f=1.75	f=2.00
c^2	G	G	G	G
0.0	3.000	3.000	3.000	3.000
0.1	2.995	2.996	2.996	2.996
0.2	2.991	2.991	2.992	2.993
0.3	2.986	2.987	2.988	2.989
0.4	2.982	2.984	2.985	2.986
0.5	2.978	2.980	2.982	2.984
0.6	2.975	2.977	2.979	2.981
0.7	2.971	2.974	2.976	2.979
0.8	2.969	2.971	2.974	2.977
0.9	2.966	2.970	2.973	2.976
1.0	2.967	2.971	2.974	2.973

a=4.00	f=0.00	f=0.25	f=0.50	f=0.75	f=1.00
c^2	G	G	G	G	G
0.0	3.000	3.000	3.000	3.000	3.000
0.1	2.994	2.995	2.995	2.995	2.995
0.2	2.989	2.989	2.990	2.991	2.991
0.3	2.983	2.984	2.985	2.986	2.987
0.4	2.978	2.979	2.981	2.982	2.983
0.5	2.973	2.975	2.976	2.978	2.979
0.6	2.968	2.970	2.972	2.974	2.976
0.7	2.964	2.966	2.968	2.970	2.972
0.8	2.960	2.962	2.965	2.967	2.969
0.9	2.956	2.959	2.962	2.964	2.967
1.0	2.955	2.958	2.961	2.964	2.967

a=4.00	f=1.25	f=1.50	f=1.75	f=2.00
c^2	G	G	G	G
0.0	3.000	3.000	3.000	3.000
0.1	2.996	2.996	2.996	2.997
0.2	2.992	2.992	2.993	2.993
0.3	2.988	2.989	2.990	2.990
0.4	2.984	2.985	2.986	2.988
0.5	2.981	2.982	2.983	2.985
0.6	2.977	2.979	2.981	2.983
0.7	2.974	2.976	2.978	2.980
0.8	2.972	2.974	2.976	2.979
0.9	2.970	2.972	2.975	2.977
1.0	2.970	2.973	2.976	2.979